that is,

$$u_{n,l,m}(r,\theta,\varphi)=\sum_{n_1=0}^{n-m-1}\langle l|n_1\rangle u_{n_1,n_2,m}(\xi,\eta,\varphi). \quad \text{(A1)}$$

The spherical and parabolic coordinate systems are related by

$$\begin{aligned}\xi&=r(1+\cos\theta),\\ \eta&=r(1-\cos\theta),\\ \varphi&=\varphi.\end{aligned} \quad \text{(A2)}$$

By inserting the explicit forms of the wave functions in terms of Laguerre polynomials and Legendre functions[25] in (A1), substituting (A2) in the right-hand side, taking $\cos\theta=1$, and equating powers of r, we get the following relation:

$$(-)^{l+m}\left[\frac{(l-m)!(n-l-1)!}{(l+m)!(n+l)!}(2l+1)\right]^{\frac{1}{2}} \times\frac{2^m s!}{[s-(l-m)]!}\binom{n+l}{l+m+1+s}\frac{d^mP_l(1)}{dz^m}$$
$$=\sum_{n_1=s}^{n-m-1}\left[\frac{n_1!n_2!}{(n_1+m)!(n_2+m)!}\right]^{\frac{1}{2}} \times\binom{n_1+m}{s+m}\binom{n_2+m}{m}\langle l|n_1\rangle. \quad \text{(A3)}$$

That the left-hand side is zero for $s<l-m$, is to be understood. Equation (A3) is used as a recursion relation to derive all of the coefficients $\langle l|n_1\rangle$.

[25] H. A. Bethe and E. E. Salpeter, reference 12, Sec. 3 and 6.

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Meromorphic Property of the S Matrix in the Complex Plane of Angular Momentum

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A proof is given for the meromorphic nature of the S matrix in the entire complex plane of the angular momentum, under quite general assumptions for the potential. Some properties for the S matrix in the complex angular momentum plane are discussed.

THE S matrix for the Schrödinger equation has been shown by Regge[1] to be meromorphic in the complex plane of the angular momentum ν to the right of the line $\mathrm{Re}\nu=\max(-3/2, -1/2-c)$, if the potential $V(x)\to V_0x^{2c-2}$ as $x\to 0$. We here propose a method to enable us to establish the meromorphic property of the S matrix for the Schrödinger equation in the whole ν plane.

The Schrödinger equation is

$$[(d^2/dx^2)-\nu(\nu+1)/x^2]\psi(k,\nu,x)=U(x)\psi(k,\nu,x), \quad (1)$$

where

$$U(x)\equiv V(x)-k^2,$$

with the boundary condition

$$\psi(k,\nu,x)\to x^{\nu+1}, \quad x\to 0.$$

Regge transformed this differential equation into an integral equation, with its solution obtained from iteration

$$\psi(k,\nu,x)=x^{\nu+1}+\int_0^x\frac{(x^{\nu+1}/y^\nu-y^{\nu+1}/x^\nu)}{(2\nu+1)}U(y)\psi(k,\nu,y)dy.$$

The iteration process fails if the integrals in the iteration diverge ($\int_0^x y^a dy$ diverges if $\mathrm{Re}a\leqslant -1$) and some integrals in every order of the iteration were indeed found to diverge for $\mathrm{Re}\nu\leqslant\max(-3/2, -1/2-c)$.

Instead, we define a linear operator K_ν

$$K_\nu(x^p)\equiv x^{p+2}/(p+\nu+2)(p-\nu+1); \quad (2)$$

we are guided by the fact that

$$\int_0^x\frac{x^{\nu+1}/y^\nu-y^{\nu+1}/x^\nu}{2\nu+1}y^pdy=\frac{x^{p+2}}{(p+2+\nu)(p-\nu+1)},$$

if the integral does not blow up at the lower limit of integration. Now, $K_\nu(f(x))$ is defined if $f(x)$ is a sum of terms x^p, or an infinite, absolutely convergent series of terms x^p. The power p does not have to be an integer; in fact, it does not even have to be real. As

$$\frac{d^2}{dx^2}K_\nu(x^p)=x^p+\nu(\nu+1)/x^2K_\nu(x^p); \quad (3)$$

we have, in general, if $K_\nu(f(x))$ exists,

$$\frac{d^2}{dx^2}K_\nu(f(x))=f(x)+\nu(\nu+1)/x^2K_\nu(f(x)).$$

[1] T. Regge, Nuovo cimento **14**, 951 (1959), also Nuovo cimento **18**, 947 (1960).

One then readily proves that the solution of Eq. (1) can be expressed as

$$\psi(k,\nu,x)=x^{\nu+1}+K_\nu(U(x)\psi(k,\nu,x)). \tag{4}$$

Now Eq. (4) can be iterated, without any divergence difficulty.

As an example, we first choose $U(x)=V_0x^p$ where[2] $\mathrm{Re}p>-2$. Then we have

$$\psi(k,\nu,x)=x^{\nu+1}\sum_{n=0}^{\infty}\frac{[V_0x^{p+2}/(p+2)]^n}{n!\prod_{m=1}^{n}[m(p+2)+2\nu+1]}. \tag{5}$$

The terms in the infinite sum are meromorphic in ν with simple poles at $\nu=-1/2[1+m(p+2)]$, $m=1, 2, 3, \cdots$. If we keep away from these poles we see that the series is uniformly convergent. Therefore, $\psi(k,\nu,x)$ is meromorphic in the whole ν plane, with only simple poles at the locations mentioned above. For $p=0(2c-2)$, the pole nearest to the origin is $\nu=-3/2(-1/2-c)$, and that is why the old method fails in the portion of the plane left to these points. If $U(x)=V_1x^{p_1}+V_2x^{p_2}$, the poles would occur at $\nu=-1/2[1+m_1(p_1+2)+m_2(p_2+2)]$, with m_1, m_2 positive integers, and $m_1+m_2=1, 2, 3, \cdots$. If $U(x)$ is an absolutely convergent series

$$\sum_{i=0}^{\infty}a_ix^{p_i},$$

the poles would occur at

$$\nu=-1/2[1+\sum_{i=0}^{\infty}m_i(p_i+2)],$$

with all the m_i positive integers, and

$$\sum_{i=0}^{\infty}m_i=1,2,3,\cdots.$$

[2] If $\mathrm{Re}p<-2$, then the behavior of the wave function near the origin will be dominated by the potential instead of by the centrifugal force. We do not consider such cases.

The iteration series still converge uniformly for all finite x. In particular, for the Yukawa potential, the poles would be at $\nu=-1,\ -3/2-2-5/2\cdots$. The poles in $\psi(k,\nu,x)$ are determined by the potential alone, independent of the energy. Now

$$\psi(k,\nu,x)=[f(k,\nu)f(-k,\nu,x)-f(-k,\nu)f(k,\nu,x)]/(2ik), \tag{6}$$

where $f(\pm k,\nu,x)$ are solutions of Eq. (1) with the boundary condition $f(\pm k,\nu,x)\rightarrow e^{\mp ikx}$ as $x\rightarrow\infty$, and are known to be entire in ν. The Jost functions,

$$f(\pm k,\nu)=f(\pm k,\nu,x)\frac{d}{dx}\psi(k,\nu,x)-\frac{df(\pm k,\nu,x)}{dx}\psi(k,\nu,x),$$

in general, would both have a pole when $\psi(k,\nu,x)$ has a pole. Therefore, the scattering matrix $S(k,\nu)=[f(k,\nu)/f(-k\nu)]e^{i\pi\nu}$, in general, has no pole when $\psi(k,\nu,x)$ has a pole. We see that the Regge poles do not come from the poles of $\psi(k,\nu,x)$. The S matrix is, therefore, meromorphic in ν, with poles arising from the zeroes of $f(-k,\nu)$.

For a Yukawa potential, $V(x)=V_0e^{-\mu x}/x$, Eq. (1) can be written as

$$\left[\frac{d^2}{dy^2}-\frac{\nu(\nu+1)}{y^2}\right]\psi(k,\nu,y)=\left[\frac{V_0}{k}\frac{e^{-\mu y/k}}{y}-1\right]\psi(k,\nu,y), \tag{7}$$

with

$$y\equiv kx.$$

Since $e^{-(\mu y/x)}\rightarrow 1$ as $k\rightarrow\infty$, the Yukawa potential approaches the Coulomb potential in the high-energy limit, positive or negative, and we see that for any point in the ν plane the scattering matrix $S(k,\nu)$ for the Yukawa potential in the high-energy limit approaches $\Gamma(\nu+1+iV_0/\sqrt{E})/\Gamma(\nu+1-iV_0/\sqrt{E})$, the scattering matrix for the Coulomb potential. In particular there are Regge poles approaching the negative integer points of the ν plane, with their residues approaching zero.

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